Solving Boundary Value Problems on Networks using Equilibrium Measures

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Abstract

The purpose of this paper is to construct the solution of self-adjoint boundary value problems on finite networks. To this end, we obtain explicit expressions of the Green functions for all different boundary value problems. The method consists in reducing each boundary value problem either to a Dirichlet problem or to a Poisson equation on a new network closely related with the former boundary value problem. In this process we also get an explicit expression of the Poisson kernel for the Dirichlet problem. In all cases, we express the Green function in terms of equilibrium measures solely, which can be obtained as the unique solution of Linear Programming Problems. In particular, we get analytic expressions of the Green function for the following problems: the Poisson equation on a distance-regular graph, the Dirichlet problem on an infinite distance-regular graph and the Neumann problem on a ball of an homogeneous tree.

1 Introduction

In this paper, we are concerned with the discrete analogues of Boundary Value Problems (BVPs) for a Laplacian operator on Riemannian manifolds. As usual, an appropriate framework to develop this work is to consider such BVPs on networks.

Two topics are closely related with the BVPs on networks: estimates of bounds for eigenvalues of a Laplacian operator,(see [5, 6]) and bound estimates of the related Green functions, (see [13]). An extensive study about Green functions on networks has been developed by M.Yamasaki and co-workers (see [9, 12, 14].) Nevertheless, few explicit expressions for the Green functions are known, see for instance the works due to P. Cartier [4] and H. Urakawa [13].

Here, we obtain the Green functions of self-adjoint BVPs on finite networks. Specifically, we study second-order partial difference equations on a subnetwork with different boundary conditions (Dirichlet, Neumann, Robin and Mixed conditions.) We also consider the limit case, that is, the Poisson equation in which the boundary of the network is empty. The Green functions for the Dirichlet and Poisson problems on a graph were obtained by the authors in [2]. In this issue, we calculate the Green functions for the BVPs by reducing them either to a Dirichlet problem or to a Poisson equation with respect to the Laplacian of a suitable network associated with the original BVP. This network consists of the initial subnetwork and of its edge and vertex boundaries. In this way, the Laplacian of the new network contains both the old Laplacian and the normal derivative on the vertex boundary (Neumann boundary condition). The relation between these tree operators is established in the Green's Identities. Our method allows to obtain the solution of Dirichlet, Neumann, Robin and Mixed Problems with non-homogeneous boundary data directly from the Green function of the new problem. In particular, we obtain an expression of the discrete version of the Poisson kernel for the Dirichlet problem and we extend the definition of such kernel to mixed problems.

The Green function of each BVP will be systematically expressed by means of equilibrium measures, which are the solution of suitable equilibrium problems in the context of the Potential Theory, considering the Laplacian as a kernel. The consideration of the Laplacian as a kernel on the vertex set of a graph, was introduced by the authors in [1, 2]. It was proved there that the Laplacian kernel verifies the maximum and energy principles which allowed to conclude that the so-called Equilibrium Problem has a unique solution for every proper subset of vertices. In addition, the equilibrium measures for such subsets can be obtained as the solution of Linear Programming Problems in which the Laplacian acts as the coefficient matrix of the general linear constraints. The extension of this method to the context of networks used in this paper is straightforward. When the network has a high degree of symmetry, the equilibrium measures can be computed by hand and so are the Green functions. This is the case of distance-regular graphs. In particular, for this type of graphs, we construct the Green function for the Poisson equation. Also, for the

Dirichlet problem on a ball, we calculate the Green function with pole in the center of the ball. Then, taking limit with respect to the radius, we deduce the Green function for an infinite distance-regular graph. Finally, we obtain the Green function for the Neumann problem on a ball of an homogeneous tree.

2 Preliminaries

Let V be a finite space with n points, F a non-empty subset of V and we suppose that V is endowed with the discrete topology. Then, the set of functions on V, denoted by $\mathcal{C}(V)$, and the set of non-negative functions on V, $\mathcal{C}^+(V)$, are naturally identified with $\mathbb{R}^n$ and the positive cone of $\mathbb{R}^n$, respectively. If $u \in \mathcal{C}(V)$, its *support* is given by $S(u) = \{x \in V : u(x) \neq 0\}$. Moreover, we consider the sets $\mathcal{C}(F) = \{u \in \mathcal{C}(V) : S(u) \subset F\}$ and $\mathcal{C}^+(F) = \mathcal{C}(F) \cap \mathcal{C}^+(V)$. A symmetric function $\mathcal{K} : V \times V \longrightarrow \mathbb{R}$ will be called a *kernel* on V. Clearly, a kernel on V is identified with a real symmetric matrix of order n.

On the other hand, the set of Radon measures on V, denoted by $\mathcal{M}(V)$, is identified with $\mathcal{C}(V)$ and hence, if $\mu \in \mathcal{M}(V)$, its *support* is defined as above. Therefore, the sets of Radon measures supported by F, $\mathcal{M}(F)$, and positive Radon measures supported by F, $\mathcal{M}^+(F)$, are identified with $\mathcal{C}(F)$ and $\mathcal{C}^+(F)$, respectively. In addition, if $\mu \in \mathcal{M}(V)$ its *mass* is given by $||\mu|| = \sum\limits_{x\in V} |\mu(x)|$ and we denote by $\mathcal{M}^1(F)$, the set of positive Radon measures supported by F with unit mass. Finally, for each $x \in V$, ε_x stands for the Dirac measure on x, whereas the measure $\sum\limits_{x\in F} \varepsilon_x$ will be denoted by $\mathbf{1}_F$.

In this paper the set V will be the set of vertices of a connected *electrical network* $\Gamma = (V, E, c)$, that is, a simple and finite connected graph, with vertex set V and edge set E, in which each edge (x, y) has been assigned a *conductance* $c(x, y) > 0$. We say that x is adjacent to y, $x \sim y$, if $(x, y) \in E$. The *degree* of x, $k(x)$, is the number of vertices adjacent to x. Given $F \subset V$, we denote by F^c its complementary in V and we consider the subsets $\partial(F) = \{(x, y) \in E : x \in F, y \in F^c\}$, called *edge boundary* of F, $\delta(F) = \{x \in F^c : (x, y) \in E \text{ for some } y \in F\}$, called *vertex boundary* of F and $\bar{F} = F \cup \delta(F)$.

The fundamental kernel in this work will be the *Laplacian* of Γ, that is, the kernel defined by $\mathcal{L}(x, y) = -c(x, y)$ if $x \sim y$, $\mathcal{L}(x, x) = \sum\limits_{y\sim x} c(x, y)$ and $\mathcal{L}(x, y) = 0$ otherwise. We are concerned with two points of view of the Laplacian of Γ: on the one hand, it is considered as the discrete version of an elliptic operator and, on the other hand, it is viewed as a kernel on a finite space in Potential Theory. In the first case we are devoted to raise and solve boundary value problems related with this operator. Note that for every $u \in \mathcal{C}(V)$, $\mathcal{L}u$ can be seen as $\mathrm{div}(\mathrm{C}\ \nabla u)$, where C is the diagonal matrix of the conductance on the edges of Γ, (see [3, 11].) In the second case, we study the equilibrium problems for the subsets of V. To do this, we take advantage of the properties of this type of kernels, (see [1, 2].) Both points of view meet here since the solution of suitable equilibrium problems enables us to obtain the solution of BVPs. The Green functions are the meeting point of both problems because these functions can be systematically expressed by means of equilibrium measures.

We also consider kernels on V defined by $\mathcal{K}(x,y) = \mathcal{L}(x,y)$ if $x \neq y$ and $\mathcal{K}(x,x) = \mathcal{L}(x,x) + q(x)$, where $q \in \mathcal{C}^+(V)$. If $\mu \in \mathcal{M}(V)$, the *potential of* μ with respect to $\mathcal{K}$, is given by

$$\mathcal{K}\mu(x) = \sum_{y \sim x} c(x,y)\Big(\mu(x) - \mu(y)\Big) + q(x)\mu(x),$$

and the *energy of* μ with respect to $\mathcal{K}$, is the value

$$I(\mu) = \langle \mathcal{K}\mu, \mu\rangle = \sum_{(x,y)\in E} c(x,y)\Big(\mu(x) - \mu(y)\Big)^2 + \sum_{x\in V} q(x)\mu^2(x),$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$. It is clear that $I(\mu) \geq 0$ for all $\mu \in \mathcal{M}(V)$ and $I(\mu) = 0$ iff either $\mu = a\mathbf{1}_V$, $a \in \mathbb{R}$, when $q \equiv 0$ or $\mu = 0$ otherwise.

The *Equilibrium Problem* for $F \subset V$ with respect to $\mathcal{K}$ consists of finding a positive measure, $\nu^F \in \mathcal{M}^+(F)$, such that $\mathcal{K}\nu^F(x) = 1$ for all $x \in F$.

To end this section, we present the results related to the existence, uniqueness and effective computation of the equilibrium measures. As the proof of these results are totally analogous to the ones in [1, 2] for the equilibrium problem with respect to the Laplacian of a graph, we give a sketch of the proof.

Proposition 2.1 *Let F be a non-empty subset of V. There exists a unique equilibrium measure for F, ν^F, except when $q \equiv 0$ and $F = V$, simultaneously. Moreover, $S(\nu^F) = F$ and $\nu^F = I(F)^{-1}\sigma^F$ where $(I(F), \sigma^F)$ is the solution of the following Linear Programming Problem:*

$$\begin{gathered} \min\ a \\ 0 \leq \sigma \leq \mathbf{1}_F \\ ||\sigma|| = 1 \\ \mathcal{K}\sigma \leq a\mathbf{1}_F \end{gathered}$$

Proof. Consider $\sigma \in \mathcal{M}^1(F)$. If $\mathcal{K}\sigma(x) = a$, $a \in \mathbb{R}$, for all $x \in F$, then $I(\sigma) = a \geq 0$. Let us show that if $\mathcal{K}\sigma(x) \geq I(\sigma)$ for all $x \in F$, then $\mathcal{K}\sigma(x) = I(\sigma)$ for all $x \in F$. Suppose that $\mathcal{K}\sigma(x) - I(\sigma) \geq 0$ for all $x \in F$, then

$$0 \leq \sum_{x\in S(\sigma)} (\mathcal{K}\sigma(x) - I(\sigma))\sigma(x) = \langle \mathcal{K}\sigma, \sigma\rangle - I(\sigma) = 0,$$

which implies that $\mathcal{K}\sigma(x) = I(\sigma)$ for all $x \in S(\sigma)$. Therefore, $\mathcal{K}\sigma(x) = I(\sigma)$ for all $x \in F$ and $S(\sigma) = F$, since $\mathcal{K}\sigma(x) \leq 0$ for all $x \in S^c(\sigma)$. In addition, $\mathcal{K}\sigma(x) \geq I(\sigma)$ for all $x \in F$

is equivalent to $\langle \mathcal{K}\sigma, \mu - \sigma \rangle \geq 0$ for all $\mu \in \mathcal{M}^1(F)$. But, this last condition is the Euler inequality relative to the minimization problem

$$\min_{\mu \in \mathcal{M}^1(F)} I(\mu).$$

So, as I is strictly convex on $\mathcal{M}^1(F)$, σ verifies the Euler inequality iff I attains its minimum value on $\mathcal{M}^1(F)$ at σ. Furthermore, the extremal measure is unique.

On the other hand, $I(\sigma) = \max\limits_{x \in F} \mathcal{K}\sigma(x) \geq \min\limits_{\mu \in \mathcal{M}^1(F)} \max\limits_{x \in F} \mathcal{K}\mu(x)$. Conversely, let $\mu \in \mathcal{M}^1(F)$ and consider $b = \max\limits_{x \in F} \mathcal{K}\mu(x)$. Then, $I(\mu) = \langle \mathcal{K}\mu, \mu \rangle \leq b$ which implies that $I(\sigma) \leq b$ and, *a fortiori*, $I(\sigma) \leq \min\limits_{\mu \in \mathcal{M}^1(F)} \max\limits_{x \in F} \mathcal{K}\mu(x)$.

Definitely, for each $F \subset V$, the problem $\min\limits_{\mu \in \mathcal{M}^1(F)} \max\limits_{x \in F} \mathcal{K}\mu(x)$ has as sole solution the unique measure $\sigma^F \in \mathcal{M}^1(F)$ whose potential is constant on F. To finish, it is suffices to note that $I(\sigma^F) > 0$ except when $q \equiv 0$ and $F = V$ simultaneously. ∎

3 Boundary Value Problems and Green Functions

In this section we firstly obtain the explicit solution of self-adjoint BVPs on networks in terms of its Green functions. Secondly, we express such functions by means of suitable equilibrium measures.

Next, we describe such problems:
Let $F \subset V$ with vertex boundary $\delta(F) = H_1 \cup H_2$ where $H_1 \cap H_2 = \emptyset$. Consider, $f \in \mathcal{C}(F)$, $q \in \mathcal{C}^+(F)$, $g_1 \in \mathcal{C}(H_1)$, $h \in \mathcal{C}^+(H_1)$ and $g_2 \in \mathcal{C}(H_2)$. A *Boundary Value Problem* on F consists of finding $u \in \mathcal{C}(\bar{F})$ such that

$$\left.\begin{aligned} \mathcal{L}u(x) + q(x)u(x) &= f(x), \quad && x \in F \\ \frac{\partial u}{\partial \eta}(x) + h(x)u(x) &= g_1(x), \quad && x \in H_1 \\ u(x) &= g_2(x), \quad && x \in H_2 \end{aligned}\right\} \tag{1}$$

where $\dfrac{\partial u}{\partial \eta}(x) = \sum\limits_{\substack{y \sim x \\ y \in F}} c(x,y)\Big(u(x) - u(y)\Big)$ is the discrete analogue of the *normal derivative* of u, (see [5].)

Problem (1), known as *Mixed problem (Robin-Dirichlet)*, summarizes the different boundary value problems that appear in the literature with proper name:

(i) *Poisson equation:* $F = V$.

(ii) *Dirichlet problem:* $H_1 = \emptyset$.

(iii) *Robin problem:* $H_2 = \emptyset$.

(iv) *Neumann problem:* $H_2 = \emptyset$ and $h \equiv 0$.

(v) *Mixed problem (Neumann-Dirichlet):* $h \equiv 0$.

Problems (i) and (ii) were studied by the authors in [2], in the case $q \equiv 0$ and $c(x, y) = 1$ if $x \sim y$. Its generalization to networks will play an essential role in this work.

Suppose that the subnetwork induced by F, $\langle F \rangle$, has m connected components. If we denote by F_j the vertex set of the j-th connected component, then the solution of (1) is obtained by *superposition* of the solutions of problems of type (1) on each one of the sets F_j. So, without loss of generality, we will assume that $\langle F \rangle$ is connected.

When f, g_1 and g_2 are null, problem (1) is called *homogeneous problem*. Moreover, the following *semi-homogeneous problems* are associated with problem (1):

$$\left.\begin{aligned} \mathcal{L}u(x) + q(x)u(x) = f(x), &\quad x \in F \\ \frac{\partial u}{\partial \eta}(x) + h(x)u(x) = 0, &\quad x \in H_1 \\ u(x) = 0, &\quad x \in H_2 \end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned} \mathcal{L}u(x) + q(x)u(x) = 0, &\quad x \in F \\ \frac{\partial u}{\partial \eta}(x) + h(x)u(x) = g_1(x), &\quad x \in H_1 \\ u(x) = 0, &\quad x \in H_2 \end{aligned}\right\} \tag{3}$$

and

$$\left.\begin{aligned} \mathcal{L}u(x) + q(x)u(x) = 0, &\quad x \in F \\ \frac{\partial u}{\partial \eta}(x) + h(x)u(x) = 0, &\quad x \in H_1 \\ u(x) = g_2(x), &\quad x \in H_2 \end{aligned}\right\} \tag{4}$$

As these problems are linear, if u_1, u_2 and u_3 are solutions of (2), (3) and (4) respectively, then $u = u_1 + u_2 + u_3$ is a solution of problem (1).

The key idea to study the existence and uniqueness of solutions of problem (1), will be to consider a new network built from the subnetwork induced by F adding its edge and vertex boundaries. The Laplacian of this network appears in a natural way in the discrete version of the *Green's Identities* that we develop next. For different approaches of these formulas see [6, 7, 10].

Given F, we define the network $\bar{\Gamma}(F) = (\bar{W}, \bar{E}, \bar{c})$, where $\bar{W} = \bar{F}$, $\bar{E} = \{(x, y) \in E : x \in F\}$ and the conductance function, $\bar{c}$, is the restriction of c to $\bar{E}$. We denote the

Laplacian of this network by $\bar{\mathcal{L}} = \mathcal{L}(\bar{\Gamma})$. In the sequel, we will use the natural identification between $\mathcal{C}(\bar{W})$ and $\mathcal{C}(\bar{F})$.

Proposition 3.1 *Let $F \subset V$ and $u, v \in \mathcal{C}(\bar{F})$. Then, it is verified*

(i) First Green's Identity

$$\sum_{(x,y)\in\bar{E}} c(x,y)(u(x)-u(y))(v(x)-v(y)) = \sum_{x\in F} \mathcal{L}u(x)v(x) + \sum_{x\in\delta(F)} \frac{\partial u}{\partial\eta}(x)v(x).$$

(ii) Second Green's Identity

$$\sum_{x\in F} \Big(\mathcal{L}u(x)v(x) - \mathcal{L}v(x)u(x)\Big) = \sum_{x\in\delta(F)} \Big(\frac{\partial v}{\partial\eta}(x)u(x) - \frac{\partial u}{\partial\eta}(x)v(x)\Big).$$

Proof. It suffices to observe that $\sum\limits_{(x,y)\in\bar{E}} c(x,y)(u(x)-u(y))(v(x)-v(y)) = \sum\limits_{x\in\bar{F}} \bar{\mathcal{L}}u(x)v(x)$ and $\bar{\mathcal{L}}u(x) = \mathcal{L}u(x)$ if $x \in F$ and $\bar{\mathcal{L}}u(x) = \frac{\partial u}{\partial\eta}(x)$ if $x \in \delta(F)$. ∎

Corollary 3.2 *Problem (1) is formally self-adjoint.*

Proof. From the second Green's identity it is verified that $\sum\limits_{x\in F}\mathcal{L}u(x)v(x) = \sum\limits_{x\in F}\mathcal{L}v(x)u(x)$, for all $u, v \in \{w \in \mathcal{C}(\bar{F}) : \dfrac{\partial w}{\partial\eta}(x) + h(x)w(x) = 0, x \in H_1$ and $w(x) = 0, x \in H_2\}$. ∎

At the sight of the proof of the first Green's identity we establish the following boundary value problem on $\bar{W}$: Find $u \in \mathcal{C}(\bar{W})$ such that

$$\left.\begin{aligned} \bar{\mathcal{L}}u(x) + \bar{q}(x)u(x) &= \bar{f}(x), \quad && x \in F \cup H_1 \\ u(x) &= g_2(x), && x \in H_2 \end{aligned}\right\} \tag{5}$$

where $\bar{q} = q + h$ and $\bar{f} = f + g_1$.

The following result, whose proof is straightforward, allows us to reduce problem (1) to problem (5), i.e., to a Poisson equation or to a Dirichlet problem. We must observe that if (1) is a Poisson or Dirichlet problem, then problems (1) and (5) are the same.

Lemma 3.3 *A function $u \in \mathcal{C}(\bar{F})$ is a solution of (1) iff it is a solution of (5).*

Let us examine the existence and uniqueness of solution of problem (5). As usual, the first step consists of transforming it into a semi-homogeneous problem. Specifically, u is a solution of (5) iff $v = u - g_2$ is a solution of

$$\left.\begin{aligned} \bar{\mathcal{L}}v(x) + \bar{q}(x)v(x) = \hat{f}(x), &\quad x \in F \cup H_1 \\ v(x) = 0, &\quad x \in H_2 \end{aligned}\right\} \tag{6}$$

where $\hat{f} = \bar{f} - \bar{\mathcal{L}}g_2 - \bar{q}g_2 = \bar{f} - \bar{\mathcal{L}}g_2$, because functions g_2 and $\bar{q}$ have disjoint supports.

Proposition 3.4 *If $H_2 = \emptyset$, $q \equiv 0$ and $h \equiv 0$, problem (1) has solution iff $\sum\limits_{x\in F} f(x) + \sum\limits_{x\in H_1} g_1(x) = 0$. Moreover, the solution is unique up to a constant. Otherwise, problem (1) has a unique solution.*

Proof. Let $w \in \mathcal{C}(F \cup H_1)$ such that $\bar{\mathcal{L}}w(x) + \bar{q}(x)w(x) = 0$. Then, $\langle \bar{\mathcal{L}}w, w\rangle + \langle \bar{q}w, w\rangle = 0$ and hence, $w = a\mathbf{1}_{\bar{F}}$, $a \in \mathbb{R}$ when $\bar{q} \equiv 0$ and $H_2 = \emptyset$, and $w \equiv 0$ otherwise. Therefore, the result follows by applying the Fredholm alternative to problem (6) and from Lemma 3.3. ∎

It has to be noticed that when the solution of (1) is unique, it can be obtained by superposition of the unique solutions of problems (2), (3) and (4). In addition, the sole BVPs whose solution is non unique, are the Poisson equation and the Neumann problem when $q \equiv 0$. For the Neumann problem, the solutions can be obtained by superposition of solutions of problems (2) and (3) iff $\sum\limits_{x\in F} f(x) = \sum\limits_{x\in\delta(F)} g_1(x) = 0$.

On the other hand, the solution of problem (2) can be expressed by means of its Green function. A function $G : \bar{F} \times F \longrightarrow \mathbb{R}$ is called the *Green Function* of the BVP (2) iff for all $y \in F$, $G_y = G(\cdot, y)$ verifies the following properties:

$$\left.\begin{aligned} \mathcal{L}G_y(x) + q(x)G_y(x) = \varepsilon_y(x) - a\mathbf{1}_F(x), &\quad x \in F \\ \frac{\partial G_y}{\partial \eta}(x) + h(x)G_y(x) = 0, &\quad x \in H_1 \\ G_y(x) = 0, &\quad x \in H_2 \\ a\langle G_y, \mathbf{1}_F\rangle = 0, & \end{aligned}\right\} \tag{7}$$

where $a = \frac{1}{|F|}$ when $H_2 = \emptyset$, $q \equiv 0$ and $h \equiv 0$, and $a = 0$ otherwise.

The following result summarizes some basic facts about the Green function.

Proposition 3.5 *There exists a unique Green function for problem (2). Besides, it is symmetric on F and when (2) has solution, the function $u(x) = \sum\limits_{y\in F} G(x,y)f(y)$ is a* *solution. Moreover, when the homogeneous problem has non trivial solution, then u is the unique solution orthogonal to $\mathbf{1}_F$.*

Proof. The existence and uniqueness of G_y, for all $y \in F$, is derived from Proposition 3.4, since in the case that the homogeneous problem has non trivial solution it is verified that

$$\sum_{x\in F}\Big(\varepsilon_y(x) - \frac{1}{|F|}\mathbf{1}_F(x)\Big) = 0 \ \text{ and } \ \langle G_y, \mathbf{1}_F\rangle = 0.$$

Given $x, y \in F$, it is verified that $\sum_{z\in F} \mathcal{L}G_x(z)G_y(z) = \sum_{z\in F} \mathcal{L}G_y(z)G_x(z)$, from Corollary 3.2 and hence,

$$G(x,y) = \sum_{z\in F}\Big(\mathcal{L}G_x(z) + q(z)G_x(z)\Big)G_y(z) = \sum_{z\in F}\Big(\mathcal{L}G_y(z) + q(z)G_y(z)\Big)G_x(z) = G(y,x).$$

Finally, it is clear that u is a solution of (2). Moreover, when the homogeneous problem has a non trivial solution, $\langle u, \mathbf{1}_F\rangle = \sum_{x\in F}\sum_{y\in F} G_y(x)f(y) = \sum_{y\in F} f(y)\langle G_y, \mathbf{1}_F\rangle = 0.$ ∎

The following result presents an expression of the solution of problem (1) using the Green function for problem (6), whose existence follows by applying the above proposition to problem (6) on the new network, $\bar{\Gamma}(F)$.

Proposition 3.6 *Let $\bar{G} : \bar{F} \times (F \cup H_1) \longrightarrow \mathbb{R}$ the Green function for problem (6) and suppose that the condition for the existence of solution is verified, if necessary. Then, a solution of (1) is given by*

$$u(x) = \sum_{y\in F} \bar{G}(x,y)f(y) + \sum_{y\in H_1} \bar{G}(x,y)g_1(y) + \sum_{y\in H_2}\Big(\varepsilon_x(y) - \frac{\partial}{\partial \eta_y}\bar{G}(x,y)\Big)g_2(y). \quad (8)$$

Proof. From Proposition 3.5, the function $v(x) = \sum_{y\in F\cup H_1} \bar{G}(x,y)\hat{f}(y)$ is a solution of (6). Therefore, $u(x) = \sum_{y\in F\cup H_1} \bar{G}(x,y)\hat{f}(y) + g_2(x)$ is a solution of (5) and from Lemma 3.3, it is a solution of (1).

Function u can be re-written as

$$u(x) = \sum_{y\in F} \bar{G}(x,y)f(y) + \sum_{y\in H_1} \bar{G}(x,y)g_1(y) - \sum_{y\in F} \bar{G}(x,y)\mathcal{L}g_2(y) + g_2(x),$$

since $\bar{\mathcal{L}}g_2(y) = \mathcal{L}g_2(y)$ if $y \in F$ and $\bar{\mathcal{L}}g_2(y) = \frac{\partial g_2}{\partial \eta}(y) = 0$ if $y \in H_1$. On the other hand, by applying the second Green's Identity we get that for all $x \in \bar{F}$

$$\begin{aligned}
\sum_{y\in F} \bar{G}(x,y)\mathcal{L}g_2(y) &= \sum_{y\in F} g_2(y)\mathcal{L}\bar{G}(x,y) + \sum_{y\in\delta(F)} \Big(g_2(y)\frac{\partial}{\partial\eta_y}\bar{G}(x,y) - \frac{\partial g_2}{\partial\eta}(y)\bar{G}(x,y)\Big) \\
&= \sum_{y\in\delta(F)} \Big(g_2(y)\frac{\partial}{\partial\eta_y}\bar{G}(x,y) - \frac{\partial g_2}{\partial\eta}(y)\bar{G}(x,y)\Big) \\
&= \sum_{y\in H_2} g_2(y)\frac{\partial}{\partial\eta_y}\bar{G}(x,y),
\end{aligned}$$

since $g_2 \equiv 0$ on $F \cup H_1$, $\dfrac{\partial g_2}{\partial\eta}(y) \equiv 0$ on H_1 and $\bar{G}(x,y) = 0$ for all $y \in H_2$. ∎

We must note that each term in (8) is a solution of one of the three semihomogeneous problems associated with (1).

Corollary 3.7 *Under the hypotheses of the above proposition, it is verified that*

(i) A solution of (2) is given by

$$u_1(x) = \sum_{y\in F} \bar{G}(x,y)f(y).$$

(ii) A solution of (3) is given by

$$u_2(x) = \sum_{y\in H_1} \bar{G}(x,y)g_1(y).$$

(iii) A solution of (4) is given by

$$u_3(x) = \sum_{y\in H_2} \Big(\varepsilon_x(y) - \frac{\partial}{\partial\eta_y}\bar{G}(x,y)\Big)g_2(y).$$

The above results allow us to express the relation between the Green functions for problems (2) and (6).

Proposition 3.8 *Let $\bar{G} : \bar{F} \times (F \cup H_1) \longrightarrow \mathbb{R}$ the Green function for problem (6). Then, $G : \bar{F} \times F \longrightarrow \mathbb{R}$ defined by $G(x,y) = \dfrac{1}{|F|^2}\displaystyle\sum_{z,w\in F}\Big(\bar{G}(x,y) - \bar{G}(x,z) - \bar{G}(y,w) + \bar{G}(z,w)\Big)$ if $q \equiv 0$, $h \equiv 0$, $H_2 = \emptyset$ and $H_1 \neq \emptyset$, and by $G(x,y) = \bar{G}(x,y)$ otherwise, is the Green function for problem (2).*

Proof. When the unique solution of the homogeneous problem associated with (1) is the trivial solution, then for all $y \in F$ the Green function for (6) with pole on y is the solution of the boundary value problem (2) with $f = \varepsilon_y$. So, the result follows by applying Corollary 3.7(i). On the other hand, problems (1) and (5) are the same for the Poisson equation, and hence $G \equiv \bar{G}$. Finally, for the Neumann problem with $q \equiv 0$, for each $y \in F$, G_y is the unique solution of problem (2) with $f = \varepsilon_y - \dfrac{1}{|F|}\mathbf{1}_F$, which is orthogonal to $\mathbf{1}_F$. Therefore, by applying Corollary 3.7(i), $G(x,y) = \bar{G}(x,y) - \dfrac{1}{|F|}\sum_{z\in F}\bar{G}(x,z) + a(y)$. Finally, taking into account that $\langle G_y, \mathbf{1}_F\rangle = 0$, we get $a(y) = -\dfrac{1}{|F|}\sum_{z\in F}\bar{G}(y,z) + \dfrac{1}{|F|^2}\sum_{z,w\in F}\bar{G}(z,w)$. ∎

In the case $H_2 = \delta(F)$ and $q \equiv 0$, the solution given in Corollary 3.7(iii) is the unique harmonic function that takes the prescribed value g_2 on $\delta(F)$. Therefore, the kernel $P(x,y) = \varepsilon_x(y) - \dfrac{\partial}{\partial \eta_y}\bar{G}(x,y)$ can be considered as the discrete version of the Poisson kernel. We can go on to define the *Poisson Kernel* of problem (4) as the function $P : \bar{F} \times H_2 \longrightarrow \mathbb{R}$ given by

$$P(x,y) = \varepsilon_x(y) - \frac{\partial}{\partial \eta_y}G(x,y),$$

where G is the Green kernel for problem (2). Observe that $G \equiv \bar{G}$ on $\bar{F}\times F$, because $H_2 \neq \emptyset$, and hence $\dfrac{\partial}{\partial \eta_y}\bar{G}(x,y) = -\sum_{\substack{z\sim y\\ z\in F}} c(y,z)\bar{G}(x,z) = \dfrac{\partial}{\partial \eta_y}G(x,y)$. So, the unique solution of (4) can be re-written as $u_3(x) = \sum_{y\in H_2} P(x,y)g_2(y)$.

As shown, the Green function for problem (6) is the corner-stone of the developed theory. So, we finish this section by expressing the Green functions in terms of equilibrium measures. Specifically, we denote by $\nu^{F\cup H_1}$ and $\nu_y^{F\cup H_1}$ the equilibrium measures for the sets $F \cup H_1$ and $\big(F\cup H_1\big)\setminus\{y\}$ respectively, with respect to the kernel $\bar{\mathcal{K}}$ defined by $\bar{\mathcal{K}}(x,y) = \bar{\mathcal{L}}(x,y)$ if $x \neq y$ and $\bar{\mathcal{K}}(x,x) = \bar{\mathcal{L}}(x,x) + \bar{q}(x)$.

Proposition 3.9 *The Green function of problem (6) is given by:*

(i) $\bar{G}(x,y) = \dfrac{1}{|F\cup H_1|^2}\Big(||\nu_y^{F\cup H_1}|| - |F\cup H_1|\,\nu_y^{F\cup H_1}(x)\Big)$ *if* $H_2 = \emptyset$ *and* $\bar{q} \equiv 0$.

(ii) $\bar{G}(x,y) = \dfrac{\nu^{F\cup H_1}(x) - \nu_y^{F\cup H_1}(x)}{1 + \sum_{z\sim y} c(y,z)\nu_y^{F\cup H_1}(z)}$ *if* $H_2 \neq \emptyset$ *or* $\bar{q} \not\equiv 0$.

Proof. (i) We denote $s = |F\cup H_1|$. Firstly, note that

$$0 = \langle \bar{\mathcal{L}}\mathbf{1}_{F\cup H_1}, \nu_y^{F\cup H_1}\rangle = \langle \mathbf{1}_{F\cup H_1}, \bar{\mathcal{L}}\nu_y^{F\cup H_1}\rangle = s - 1 + \bar{\mathcal{L}}\nu_y^{F\cup H_1}(y).$$

Therefore, for all $x, y \in F \cup H_1$ we get that

$$\bar{\mathcal{L}}\Big(||\nu_y^{F\cup H_1}|| - s\nu_y^{F\cup H_1}\Big)(x) = -s\bar{\mathcal{L}}\nu_y^{F\cup H_1}(x) = \begin{cases} -s & \text{if } x \neq y, \\ s(s-1) & \text{if } x = y \end{cases}$$

and hence, $\bar{\mathcal{L}}\bar{G}_y = \varepsilon_y - \dfrac{1}{s}\mathbf{1}_{F\cup H_1}$.

Finally, $\langle \bar{G}_y, \mathbf{1}_{F\cup H_1}\rangle = \dfrac{1}{s^2}\sum\limits_{x\in F\cup H_1}\Big(||\nu_y^{F\cup H_1}|| - s\nu_y^{F\cup H_1}(x)\Big) = 0.$

(ii) For all $x, y \in F \cup H_1$ we have that

$$\bar{\mathcal{L}}\Big(\nu^{F\cup H_1} - \nu_y^{F\cup H_1}\Big)(x) = \begin{cases} 0 & \text{if } x \neq y \\ 1 + \sum\limits_{z\sim y} c(y,z)\nu_y^{F\cup H_1}(z) & \text{if } x = y \end{cases}$$

and hence, $\bar{\mathcal{L}}\bar{G}_y(x) = \varepsilon_y(x)$. Moreover, when $H_2 \neq \emptyset$, for all $y \in F \cup H_1$, $x \in H_2$ we get that $\bar{G}_y(x) = 0$ since $S(\nu^{F\cup H_1} - \nu_y^{F\cup H_1}) \subset F \cup H_1$. ∎

4 Applications

In this section we find an analytical expression of the Green function for some relevant problems on graphs. Specifically, we construct the Green function for the Poisson equation of a distance-regular graph and we calculate the Green function for the Dirichlet problem for the end compactification of an infinite distance-regular graph. Finally, we give the expression of the Green function for the Neumann problem on a ball of an homogeneous tree.

We start with some basic terminology. For any vertex $y \in V$ we denote by $\Gamma_i(y)$ the set of vertices at distance i from y and by $B_r(y)$ the ball centered at y with radius r, i.e. $B_r(y) = \bigcup\limits_{i=0}^{r}\Gamma_i(y)$. A connected graph Γ is called *distance-regular* if there are integers b_i, c_i, $i = 0, \dots, d$, such that for any two vertices $x, y \in V$ at distance $i = d(x,y)$, there are exactly c_i neighbours of x in $\Gamma_{i-1}(y)$ and b_i neighbours of x in $\Gamma_{i+1}(y)$. In particular, Γ is regular of degree $k = b_0$. The sequence

$$\iota(\Gamma) = \{b_0, b_1, \dots, b_{d-1}; c_1, \dots, c_d\},$$

is called the *intersection array* of Γ. In addition, $a_i = k - c_i - b_i$ is the number of neighbours of x in $\Gamma_i(y)$, for $d(x,y) = i$. Clearly, $b_d = c_0 = 0$, $c_1 = 1$ and the diameter of Γ is d. For any vertex $y \in V$ the number of vertices at distance i from it, i.e. $|\Gamma_i(y)|$, does not depend on the vertex y and will be denoted by k_i. Moreover, the following equalities hold:

$$k_0 = 1,\ k_1 = k,\ \ k_{i+1}c_{i+1} = k_i b_i,\ i = 0, \dots, d-1. \tag{9}$$

Clearly, in a distance-regular graph, the cardinal of $B_r(y)$ and $\partial(B_r(y))$ do not depend on y and they will be denoted by $|B_r|$ and $|\partial B_r|$, respectively.

To construct the Green function for the Poisson equation we first need to compute the equilibrium measure for the subsets $V \setminus \{y\}$, for all $y \in V$.

Proposition 4.1 *Let Γ be a distance-regular graph. Then, for all $y \in V$ the equilibrium measure for the set $V \setminus \{y\}$ is given by*

$$\nu_y^V(x) = \sum_{j=0}^{d(x,y)-1} \frac{n - |B_j|}{|\partial B_j|}.$$

Proof. Assume that the value $\nu_y^V(x)$ depends only on the distance from x to y, that is, there exists $q(i)$, $i = 1, \cdots, d$ such that $\nu_y^V(x) = q(i) \iff d(x,y) = i$. Under this hypothesis, the equilibrium system $\mathcal{L}\nu_y(x) = 1$ for all $x \in V \setminus \{y\}$ is equivalent to the system:

$$c_i\gamma(i-1) - b_i\gamma(i) = 1, \quad i = 1, \dots, d,$$

where $\gamma(i) = q(i+1) - q(i)$ and $q(0) = q(d+1) = 0$. If this system has a solution, the measure $\nu_y^V(x) = q(i) \iff d(x,y) = i$ will be the equilibrium measure for $V \setminus \{y\}$ since it is unique.

By multiplying the i-th equation by k_i and taking into account (9), we obtain that $k_j b_j \gamma(j) = \sum\limits_{i=j+1}^{d} k_i$, that is,

$$\gamma(j) = \frac{n - |B_j|}{|\partial B_j|}, \ \ j = 0, \dots, d-1.$$

To conclude, it suffices to observe that $q(i) = \sum\limits_{j=0}^{i-1} \gamma(j),\ i = 1, \dots, d.$ ∎

Proposition 4.2 *Let Γ be a distance-regular graph. Then, the Green function for the Poisson equation on Γ is given by*

$$G(x,y) = \sum_{j=d(x,y)}^{d} \frac{n - |B_j|}{n|\partial B_j|} - \sum_{j=0}^{d} \frac{|B_j|(n - |B_j|)}{n^2|\partial B_j|}.$$

Proof. From Proposition 3.9(i), we have that $G(x,y) = \frac{1}{n^2}(||\nu_y^V|| - n\nu_y^V(x))$. On the other hand, from Proposition 4.1,

$$||\nu_y^V|| = \sum_{j=1}^{d} k_j q(j) = \sum_{j=1}^{d}\sum_{i=0}^{j-1} k_j \frac{n-|B_i|}{|\partial B_i|} = \sum_{i=0}^{d-1}\sum_{j=i+1}^{d} k_j \frac{n-|B_i|}{|\partial B_i|} = \sum_{i=0}^{d} \frac{(n-|B_i|)^2}{|\partial B_i|}.$$

Therefore,

$$\begin{aligned} G(x,y) &= \sum_{j=0}^{d} \frac{(n-|B_j|)^2}{n^2|\partial B_j|} - \sum_{j=0}^{d(x,y)-1} \frac{n-|B_j|}{n|\partial B_j|} \\ &= \sum_{j=0}^{d} \frac{n-|B_j|}{n|\partial B_j|} - \sum_{j=0}^{d} \frac{|B_j|(n-|B_j|)}{n^2|\partial B_j|} - \sum_{j=0}^{d(x,y)-1} \frac{n-|B_j|}{n|\partial B_j|} \\ &= \sum_{j=d(x,y)}^{d} \frac{n-|B_j|}{n|\partial B_j|} - \sum_{j=0}^{d} \frac{|B_j|(n-|B_j|)}{n^2|\partial B_j|}. \quad \blacksquare \end{aligned}$$

This technique enables us to obtain the Green function for the Dirichlet problem for the end compactification of an infinite distance-regular graph. Such graphs have been characterized by A.A. Ivanov [8]. Specifically, there was proved that the intersection array of an infinite distance-regular graph is $c_i = 1$ and $b_i = k - l, i \geq 1$ with $k > l \geq 1$. In particular, when $l = 1$ they are homogeneous trees and when $l = k-1$, necessarily $k = 2$ and the graph is a doubly infinite path. In addition, when l divides k, such graphs exist and their Green functions for the Dirichlet problem have been obtained by H. Urakawa [13] in a different way to the one we developed here.

The achievement of the Green function is based on the *exhaustion method*, as the following result asserts, (see [13, Th. 4.6].)

Lemma 4.3 *Let Γ be a locally finite, infinite connected graph and G its Green function. Then*

$$G(x,y) = \lim_{r\to\infty} G_{B_r(z)}(x,y),$$

where $G_{B_r(z)}$ is the Green function for the Dirichlet problem on $B_r(z)$, for some fixed z.

Proposition 4.4 *Let Γ be an infinite distance-regular graph. Then its Green function is given by*

$$G(x,y) = \sum_{j=d(x,y)}^{\infty} \frac{1}{|\partial B_j|}.$$

Proof. From Lemma 4.3 it will suffice to calculate $G_{B_r(y)}(x,y),\ x\in B_r(y)$. On the other hand, from Proposition 3.9(ii)

$$G_{B_r(y)}(x,y)=\frac{\nu^{B_r(y)}(x)-\nu_y^{B_r(y)}(x)}{1+\sum\limits_{z\sim y}\nu_y^{B_r(y)}(z)}.$$

Firstly, let us calculate $\nu^{B_r(y)}$. Suppose that its values depend only on the distance to y and let $p(j)=\nu^{B_r(y)}(x)$ if $d(x,y)=j$. Then, $p(j)$, $j=0,\dots,r$ must verify the following system

$$\begin{cases} -k\phi(0)=1 \\ c_i\phi(i-1)-b_i\phi(i)=1, \quad i=1,\dots,r, \end{cases}$$

where $\phi(i)=p(i+1)-p(i)$ and $p(r+1)=0$. Reasoning as in the proof of Proposition 4.1 we obtain that

$$\phi(j)=-\frac{|B_j|}{|\partial B_j|}\ \ \text{and}\ \ p(i)=\sum_{s=i}^{r}\frac{|B_s|}{|\partial B_s|},\ j,i=0,\dots,r.$$

Then $\nu^{B_r(y)}(x)=p(d(x,y))$ is the equilibrium measure for $B_r(y)$, since it is unique. To calculate $\nu_y^{B_r(y)}$ we make the same assumption, namely $q(j)=\nu_y^{B_r(y)}(x)$ if $d(x,y)=j$. Then, $q(j)$ has to be the solution of the system

$$c_i\gamma(i-1)-b_i\gamma(i)=1,\ i=1,\dots,r,$$

where $\gamma(i)=q(i+1)-q(i)$ and $q(0)=q(r+1)=0$. Again, multiplying the i-th equation by k_i we obtain that

$$\gamma(j)=\frac{1+k\gamma(0)-|B_j|}{|\partial B_j|},\ j=0,\dots,r$$

and hence,

$$q(i)=(1+kq(1))\sum_{j=0}^{i-1}\frac{1}{|\partial B_j|}-\sum_{j=0}^{i-1}\frac{|B_j|}{|\partial B_j|},\ i=1,\dots,r+1.$$

Taking into account the condition $q(r+1)=0$, we get that

$$1+kq(1)=\frac{\sum\limits_{j=0}^{r}\dfrac{|B_j|}{|\partial B_j|}}{\sum\limits_{j=0}^{r}\dfrac{1}{|\partial B_j|}}.$$

Finally,

$$G_{B_r(y)}(x,y)=\frac{p(d(x,y))-q(d(x,y))}{1+kq(1)}=\sum_{j=d(x,y)}^{r}\frac{1}{|\partial B_j|}.\quad\blacksquare$$

Bear in mind the intersection array of an infinite distance-regular graph, we get that $|\partial B_j| = k(k-l)^j$ and hence,

$$G(x,y) = \frac{1}{k(k-l-1)(k-l)^{d(x,y)-1}}.$$

As a consequence, it follows that the doubly infinite path is the unique *recurrent* infinite distance-regular graph.

Our last goal is to calculate the Green function for the Neumann problem on a ball of an homogenous tree. Specifically, we denote by T_k an homogeneous tree of degree $k \geq 3$ and by B_r the ball of radius r and center o, a fixed vertex in $V(T_k)$. Moreover, $|y|$ will denote the value $d(y,o)$ for any vertex $y \in V(T_k)$. Nevertheless, we keep on the notation $|B_r|$ for the cardinal of B_r.

We are concerned with the Green function for the following Neumann problem:

$$\left.\begin{array}{ll} \mathcal{L}u(x) = f(x), & x \in B_r \\ \dfrac{\partial u}{\partial \eta}(x) = 0, & x \in \delta(B_r). \end{array}\right\} \tag{10}$$

Applying the results of the preceding section, this problem is equivalent to the Poisson equation on $\bar{T}_k(B_r)$. In this case, $\bar{T}_k(B_r) = \langle B_{r+1} \rangle$ and hence $V(\bar{T}_k(B_r)) = B_{r+1}$. Then, problem (10) is equivalent to

$$\bar{\mathcal{L}}u(x) = \bar{f}(x), \; x \in B_{r+1}. \tag{11}$$

From Proposition 3.8, it suffices to calculate the Green function, $\bar{G}$ for problem (11) and from Proposition 3.9 (i) this function is given by

$$\bar{G}(x,y) = \frac{1}{|B_{r+1}|^2}\Big(||\nu_y^{B_{r+1}}|| - |B_{r+1}|\,\nu_y^{B_{r+1}}(x)\Big), \; x,y \in B_{r+1}.$$

To calculate $\nu_y^{B_{r+1}}, y \in B_{r+1}$, we suppose that the following geometrical hypothesis is verified:

(H) *If $|y| = |y'|$, then $\nu_y^{B_{r+1}}(x) = \nu_{y'}^{B_{r+1}}(x')$ whenever $|x| = |x'|$ and $d(x,y) = d(x',y')$.*

Again note that if this supposition leads to a measure μ such that $\bar{\mathcal{L}}\mu(x) = 1$ for all $x \in B_{r+1} \setminus \{y\}$, then $\nu_y^{B_{r+1}} = \mu$ because of the uniqueness of the equilibrium measure. Moreover, (H) implies that $\nu_y^{B_{r+1}}(x)$ only depends on $|y|$, $|x|$ and $d(x,y)$.

Prior to calculate the equilibrium measure, let us show the scheme of $\bar{T}_k(B_r)$, Fig. 1, in which we have identified those vertices on which the measure $\nu_y^{B_{r+1}}$ takes the same value, according to hypothesis (H). Suppose that $y \in \Gamma_t(o)$, $t = 0, \ldots, r+1$ and consider

$o = y_1, y_2, \ldots, y_{t+1} = y$ the geodesic between o and y. In addition, if $x \in B_{r+1}$, we denote by $q_t(s+1, j+1)$ the value of $\nu_y(x)$, where $s = \frac{1}{2}(|x| + |y| - d(x, y))$ and $j = \frac{1}{2}(|x| - |y| + d(x, y))$. Note that j is the distance from x to the geodesic and $s+1$ is the index of the vertex of the geodesic that gives such a distance. On the other hand, if we denote by $m(s, j)$ the number of vertices with mass $q_t(s, j)$, then

$$m(s,j) = \begin{cases} 1, & s = 1, \ldots, t+1, \;\; j = 1, \\ (k-2)(k-1)^{j-2}, & s = 2, \ldots, t, \;\; j = 2, \ldots, r+3-s, \\ (k-1)^{j-1}, & s = 1 \text{ or } s = t+1, \;\; j = 2, \ldots, r+3-s. \end{cases}$$

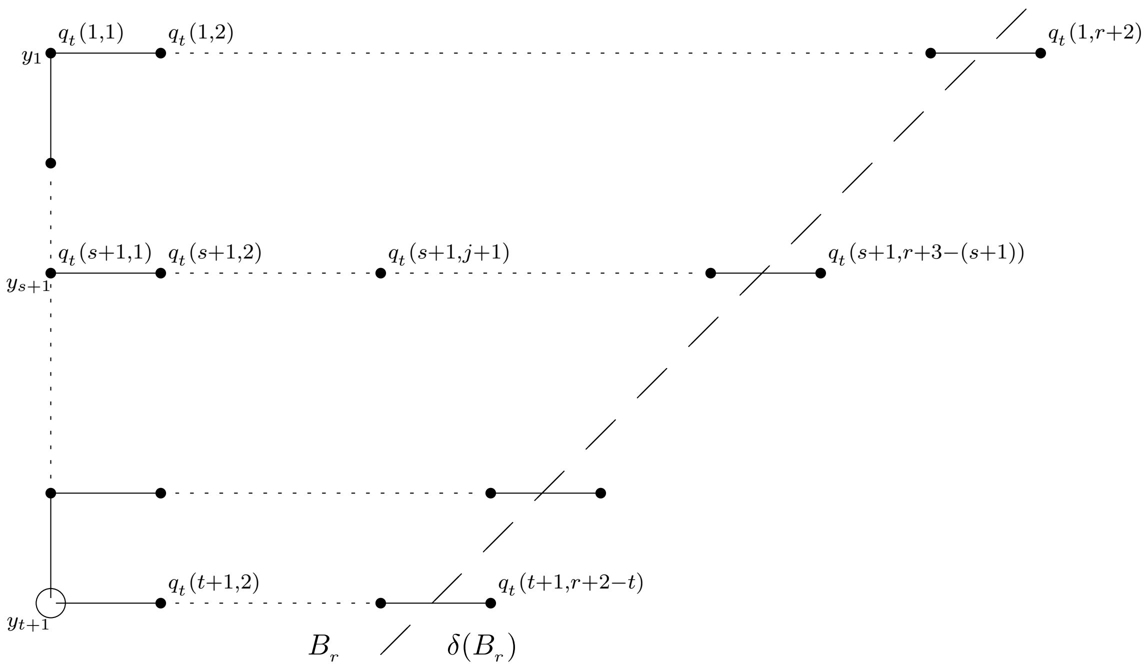

Figure 1: Mass distribution of the measure $\nu_y^{B_{r+1}}$ for B_{r+1}.

Proposition 4.5 *For all $y \in B_{r+1}$ the equilibrium measure $\nu_y^{B_{r+1}}$ for the set $B_{r+1} \setminus \{y\}$ is given by*

$$\nu_y^{B_{r+1}}(x) = \frac{k(k-1)^{r+1} - 2}{2(k-2)} d(x,y) + \frac{k(k-1)^{r+1}(|y| - |x|)}{2(k-2)} + \frac{(k-1)^{r+2-|y|} - (k-1)^{r+2-|x|}}{(k-2)^2}.$$

Proof. Under hypothesis (H) the mass distribution must verify the following systems

$$\begin{cases} kq_t(s,i) - q_t(s,i-1) - (k-1)q_t(s,i+1) = 1, & s = 1 \ldots, t+1, \; i = 2, \ldots, r+2-s, \\ q_t(s, r+3-s) - q_t(s, r+2-s) = 1, & s = 1 \ldots, t+1. \end{cases} \tag{12}$$

$$\begin{cases} kq_t(1,1) - q_t(2,1) - (k-1)q_t(1,2) = 1, \\ kq_t(s,1) - q_t(s-1,1) - q_t(s+1,1) - (k-2)q_t(s,2) = 1, \quad s = 2\dots,t-1, \\ kq_t(t,1) - q_t(t-1,1) - (k-2)q_t(t,2) = 1. \end{cases} \tag{13}$$

We must note that if $t = 0$, the system (13) has no sense and hence the mass distribution is obtained from system (12). If $t = 1$, the mass distribution is obtained from (12) and the first equation of (13). Finally, if $t = 2$, the mass distribution is obtained from (12) and the first and the last equations of (13).

If we denote by $\gamma(s,i) = q_t(s,i+1) - q_t(s,i)$, $i = 1,\dots,r+2-s$ and $\phi(s) = q_t(s+1,1) - q_t(s,1)$, $s = 1,\dots,t$, where $q_t(t+1,1) = \nu_y(y) = 0$, then systems (12) and (13) can be re-written respectively as

$$\begin{cases} \gamma(s,i-1) - (k-1)\gamma(s,i) = 1, & s = 1,\dots,t+1,\ i = 2,\dots,r+2-s, \\ \gamma(s,r+2-s) = 1, & s = 1,\dots,t+1, \end{cases} \tag{14}$$

$$\begin{cases} -\phi(1) = (k-1)\gamma(1,1) + 1, \\ \phi(s-1) - \phi(s) = (k-2)\gamma(s,1) + 1, \quad s = 2,\dots,t. \end{cases} \tag{15}$$

Therefore,

$$\gamma(s,j) = \sum_{l=0}^{r+2-s-j} (k-1)^l,\ j = 1,\dots,r+2-s$$

and

$$-\phi(s) = \sum_{l=0}^{r+1} (k-1)^l + \sum_{l=r+2-s}^{r} (k-1)^l.$$

On the other hand,

$$q_t(s,1) = -\sum_{i=s}^{t} \phi(i) = (t+1-s)\sum_{l=0}^{r+1}(k-1)^l + \sum_{i=s}^{t}\ \sum_{l=r+2-i}^{r} (k-1)^l,\ s = 1,\dots,t$$

and for all $s = 1,\dots,t+1,\ j = 2,\dots,r+3-s$,

$$\begin{aligned} q_t(s,j) &= \sum_{i=1}^{j-1} \gamma(s,i) + q_t(s,1) \\ &= (t+1-s)\sum_{l=0}^{r+1}(k-1)^l + \sum_{i=s}^{t}\ \sum_{l=r+2-i}^{r}(k-1)^l + \sum_{i=1}^{j-1}\ \sum_{l=0}^{r+2-s-i}(k-1)^l. \end{aligned}$$

Adding these expressions we conclude that

$$\begin{aligned} q_t(s,j) &= \frac{1}{k-2}\Big(k(k-1)^{r+1}(t-s+1)-(t-s+j)\Big) \\ &\quad + \frac{1}{(k-2)^2}\Big((k-1)^{r+2-t}-(k-1)^{r+4-(s+j)}\Big) \\ &= (t-(s-1))\frac{k(k-1)^{r+1}-2}{k-2}+\frac{t-(s+j-2)}{k-2} \\ &\quad + \frac{1}{(k-2)^2}\Big((k-1)^{r+2-t}-(k-1)^{r+2-(s+j-2)}\Big). \end{aligned}$$

The result follows taking into account that $t=|y|$, $s-1=\frac{1}{2}(|x|+|y|-d(x,y))$ and $s+j-2=|x|$. ∎

Corollary 4.6 *The Green function for problem (11) is given by*

$$\bar{G}(x,y) = -\frac{d(x,y)}{2}+\frac{k(k-1)^{r+1}(|x|+|y|)}{2(k(k-1)^{r+1}-2)}+\frac{(k-1)^{r+2-|x|}+(k-1)^{r+2-|y|}}{(k-2)(k(k-1)^{r+1}-2)}+\alpha,$$

where $\alpha = \dfrac{(k(r+1)-2)(k-1)^{r+2}-k(r+2)(k-1)^{r+1}+k}{2(k-2)(k(k-1)^{r+1}-2)}$.

Proof. As $\bar{G}(x,y)=\frac{1}{|B_{r+1}|^2}\Big(||\nu_y^{B_{r+1}}||-|B_{r+1}|\,\nu_y^{B_{r+1}}(x)\Big)$, $x,y\in B_{r+1}$, we first calculate the mass of the equilibrium measure, $\nu_y^{B_{r+1}}$. If $m=|B_{r+1}|=\frac{k(k-1)^{r+1}-2}{k-2}$, $\nu_y^{B_{r+1}}(x)$ can be re-written as $\nu_y^{B_{r+1}}(x)=\frac{m}{2}d(x,y)+(\frac{m}{2}+\frac{1}{k-2})(|y|-|x|)+\frac{(k-1)^{r+2-|y|}-(k-1)^{r+2-|x|}}{(k-2)^2}$. Therefore,

$$\begin{aligned} ||\nu_y^{B_{r+1}}|| &= \frac{m}{2}\sum_{x\in B_{r+1}} d(x,y)+m(\frac{m}{2}+\frac{1}{k-2})|y|+\frac{m(k-1)^{r+2-|y|}}{(k-2)^2} \\ &\quad -(\frac{m}{2}+\frac{1}{k-2})\sum_{x\in B_{r+1}}|x|-\frac{1}{(k-2)^2}\sum_{x\in B_{r+1}}(k-1)^{r+2-|x|}. \end{aligned}$$

On the other hand,

$$\sum_{x\in B_{r+1}}|x| = k\sum_{j=0}^{r+1} j(k-1)^{j-1} = \frac{k}{(k-2)^2}\Big((r+1)(k-1)^{r+2}-(r+2)(k-1)^{r+1}+1\Big)$$

and

$$\sum_{x\in B_{r+1}}(k-1)^{r+2-|x|} = (k-1)^{r+2}+k\sum_{j=1}^{r+1}(k-1)^{r+2-j}(k-1)^{j-1} = (k-1)^{r+2}+k(r+1)(k-1)^{r+1}.$$

Keeping in mind that $d(x,y) = |x| + |y| - 2(s-1)$, we obtain that

$$\begin{aligned}
\sum_{x\in B_{r+1}} d(x,y) &= m|y| + \sum_{x\in B_{r+1}} |x| - 2\sum_{s=1}^{|y|+1}(s-1)\sum_{j=1}^{r+3-s} m(s,j) \\
&= m|y| + \sum_{x\in B_{r+1}} |x| + \frac{2|y|}{k-2} + \frac{2}{(k-2)^2}\Big((k-1)^{r+2-|y|} - (k-1)^{r+2}\Big) \\
&= (m + \frac{2}{k-2})|y| + \frac{2(k-1)^{r+2-|y|}}{(k-2)^2} + \beta,
\end{aligned}$$

where $\beta = \sum_{x\in B_{r+1}} |x| - \frac{2(k-1)^{r+2}}{(k-2)^2} = \frac{(k(r+1)-2)(k-1)^{r+2} - k(r+2)(k-1)^{r+1} + k}{(k-2)^2}$.
Finaly,

$$\bar{G}(x,y) = -\frac{d(x,y)}{2} + \frac{(m(k-2)+2)}{2m(k-2)}(|x|+|y|) + \frac{(k-1)^{r+2-|y|} + (k-1)^{r+2-|x|}}{m(k-2)^2} + \frac{\beta}{2m}. \quad \blacksquare$$

Corollary 4.7 *The Green function for problem (10) is given by*

$$G(x,y) = -\frac{d(x,y)}{2} + \frac{k(k-1)^r(|x|+|y|)}{2(k(k-1)^r-2)} + \frac{(k-1)^{r+1-|x|} + (k-1)^{r+1-|y|}}{(k-2)(k(k-1)^r-2)} + \gamma,$$

where $\gamma = -\frac{k(k-1)^{2r+1} + (k-1)^{r+2} + (2kr-3)(k-1)^{r+1} - k(2r+1)(k-1)^r + k}{(k-2)(k(k-1)^r-2)^2}$.

Proof. From Proposition 3.8, the Green function for the Neumann problem is given by $G(x,y) = \frac{1}{|B_r|^2}\sum_{z,w\in B_r}\Big(\bar{G}(x,y) - \bar{G}(x,z) - \bar{G}(y,w) + \bar{G}(z,w)\Big)$, where $\bar{G}$ is the Green function for problem (11).

As $\bar{G}(x,y) - \bar{G}(x,z) - \bar{G}(y,w) + \bar{G}(z,w) = -\frac{1}{2}\Big(d(x,y) - d(x,z) - d(y,w) + d(z,w)\Big)$, we get that

$$G(x,y) = -\frac{d(x,y)}{2} + \frac{1}{2|B_r|}\sum_{z\in B_r}\Big(d(x,z) + d(y,z)\Big) - \frac{1}{2|B_r|^2}\sum_{z,w\in B_r} d(z,w).$$

Reasoning as in the proof of the above corollary we have that

$$\frac{1}{2|B_r|}\sum_{z\in B_r} d(x,z) = \frac{|B_r|(k-2)+2}{2|B_r|(k-2)}|x| + \frac{(k-1)^{r+1-|x|}}{|B_r|(k-2)^2} + \beta,$$

where $\beta = \dfrac{(kr-2)(k-1)^{r+1} - k(r+1)(k-1)^r + k}{2|B_r|(k-2)^2}$. Analogously,

$$\begin{aligned}\frac{1}{2|B_r|^2}\sum_{z,w\in B_r} d(z,w) \;=\;& \frac{|B_r|(k-2)+2}{2|B_r|^2(k-2)^3}\Big(kr(k-1)^{r+1} - k(r+1)(k-1)^r + k\Big)\\ &+\frac{(k-1)^{r+1} + kr(k-1)^r}{|B_r|^2(k-2)^2} + \beta. \quad \blacksquare\end{aligned}$$

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References

[1] E. Bendito, A. Carmona and A. M. Encinas, Shortest paths in distance-regular graphs, to appear in *European Journal of Combinatorics*.

[2] E. Bendito, A. Carmona and A. M. Encinas, Solving Dirichlet and Poisson problems on graphs by means of equilibrium measures, submitted to *Combinatorica*.

[3] I. Benjamini and O. Schramm, Harmonic functions on planar and almost planar graphs and manifolds, via circle packing, *Invent. Math.* **126** (1996), 565–587.

[4] P. Cartier, Fonctions harmoniques sur un arbre, *Sympos. Math.* **9** (1972), 203–270.

[5] F.R.K. Chung, A. Grigor'yan and S.T. Yau, Upper bounds for eigenvalues of the discrete and continuous Laplace operators, *Adv. Math.* **117** (1996), 165–178.

[6] F.R.K. Chung and S.T. Yau, Eigenvalue inequalities for graphs and convex subgraphs, *Comm. Anal. Geom.* **5** (1997), 575–623.

[7] E.B. Curtis and J.A. Morrow, Determining the resistors in a network, *SIAM J. Appl. Math.* **50** (1990), 918–930.

[8] A.A. Ivanov, Bounding the diameter of a distance-regular, *Soviet Math. Dokl.* **28** (1983), 149–152.

[9] T. Kayano and M. Yamasaki, Dirichlet finite solution of Poisson equations on a infinite network, *Hiroshima Math. J.* **12** (1982), 569–579.

[10] T. Kayano and M. Yamasaki, Discrete Dirichlet integral formula, *Discrete Appl. Math.* **22** (1988/89), 53–68.

[11] P.M. Soardi, "Potential theory on infinite networks", Lecture Notes in Mathematics 1590, Springer-Verlag (1994).

[12] P.M. Soardi and M. Yamasaki, Classification of infinite networks and its applications, *Circuits Systems Signal Process* **12** (1993), 133–149.

[13] H. Urakawa, Heat kernel and Green kernel comparison theorems for infinite graphs, *J. Funct. Anal.* **146** (1997), 206–235.

[14] M. Yamasaki, The equation $\Delta u = qu$ on an infinite network, *Mem. Fac. Sci. Shimane Univ.* **21** (1987), 31–46.